NONLINEAR STUDIES - www.nonlinearstudies.com
Vol. 19, No. 3, pp. 355-365, 2012

Some new stability and boundedness results on a certain fourth order nonlinear differential equation

Olufemi Adeyinka Adesina [1⋆], Babatunde Sunday Ogundare [2]

[1] Department of Mathematics, Obafemi Awolowo University
Ile-Ife, Nigeria
E-mail:oadesina@oauife.edu.ng

[2] Department of Mathematics, Obafemi Awolowo University
Ile-Ife, Nigeria
E-mail:bogunda@oauife.edu.ng

⋆ *Corresponding Author*. E-mail address: oadesina@oauife.edu.ng

Abstract. In this paper, some new conditions on the global asymptotic stability, boundedness and ultimate boundedness of solutions to a certain fourth order nonlinear differential equation are obtained. By constructing a complete Lyapunov function, we give three new results on the considered equation. Our results improve, generalize and complement existing results in the literature on the subject matter.

1 Introduction

The primary purpose of this paper is to study the problem of stability and boundedness of solutions to the equation

$$x^{(iv)} + \phi(\dddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}), \tag{1.1}$$

with the associated system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= -\phi(w) - f(z) - g(y) - h(x) + p(t,x,\dot{x},\ddot{x},\dddot{x}), \end{aligned} \tag{1.2}$$

where functions ϕ, f, g, h and p are continuous and depend (at most) only on the arguments displayed explicitly. Here and elsewhere, all the solutions considered and all the functions which appear are

2010 **Mathematics Subject Classification**: Primary: 34A34, 34D20, 34D23, 34D99.
Keywords: Boundedness, Ultimate boundedness, Lyapunov function, Nonlinear fourth order differential equations, Stability.

supposed real. The dots indicate differentiation with respect to t. A complete Lyapunov function is constructed for this purpose via Lyapunov's second method which is one of the most useful and widespread tool in the study of qualitative behaviour of solutions to nonlinear differential equations of higher order. It must be emphasized here that constructing a complete Lyapunov function is not an easy task.

The notions of stability and boundedness of solutions are fundamental in the theory and application of differential equations. In this way, both concepts lead to the real world applications. It is well known that the study of fourth order nonlinear differential equations has attracted the interest of many researchers. Many results relative to stability, boundedness, convergence and periodic solutions to equations of the form (1.1) have been obtained. See for instance ([1]-[26]). Adesina [2] and Afuwape [5, 8, 9] employed the frequency domain technique to obtain conditions that guarantee the existence of periodic, almost periodic, exponential stability and dissipative solutions. With the aid of Lyapunov's second method, Abou-el-ela and Sadek [1], Sadek [16], Sadek and AL-Elaiw [17] worked on asymptotic stability of solutions. In their own contributions, Chin [10], Ogundare [12], Ogundare and Okecha [14], Tiryaki and Tunc [20], Tunc [21, 22, 23, 24, 25] and Wu and Xiong [26] worked on boundedness and stability properties of solutions. An interesting work on the existence of limiting regime in the sense of Demidovich can be found in Afuwape [7].

However, as pointed out in Tejumola [18], the equation (1.1) has so far remained intractable due to the number of the nonlinear terms simultaneously involved and the form of the functions ϕ and f. In other words, results on the qualitative behaviour of solutions to the equation (1.1) are relatively scarce. Thus, it is worthwhile to investigate the stability and boundedness properties of solutions of the equation (1.1).

The motivation for the current paper comes from the works of Adesina and Ogundare [3], Afuwape [4, 6, 7] and Tejumola [18]. Our results extend the results obtained by Adesina and Ogundare [3] for the asymptotic stability of the null solution, boundedness and ultimate boundedness of all solutions when $p \equiv 0$ and $p \neq 0$ in the equation (1.1). Furthermore our results generalize the result of Ogundare and Okecha [14] and also complement existing results on the qualitative behaviour of solutions of fourth order nonlinear differential equations.

This paper is organized as follows. In Section 2, we present basic assumptions and main results. Section 3 is devoted to some preliminary results while in Section 4, we give the proofs of the main results.

2 Formulation of Results

Consider the homogeneous linear part of the equation (1.1):

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0,$$

where a, b, c and d are constants. In this case, all solutions tend to the trivial solution, as $t \to \infty$, provided that the Routh-Hurwitz criteria

$$a > 0, \quad ab - c > 0, \quad (ab - c)c - a^2 d > 0, d > 0$$

are satisfied.

Now let the functions $\phi, f,\ g,\ h$ and p be continuous and the following conditions hold for some positive constants a_0, b_0, Δ_0, Δ_1, $K_0{}^*$ and $0 < K_1{}^* < 1$:

(i)
$$a \le \frac{\phi(w)-\phi(0)}{w} = \Phi_0 \le a_0 = \frac{\varepsilon+\delta(1-\varepsilon)D}{D}, \quad (w \neq 0 \text{ and } \phi(0)=0);$$

(ii)
$$b \le \frac{f(z)-f(0)}{z} = F_0 \le b_0 = \frac{\varepsilon+\gamma\delta(1-\varepsilon)D}{\delta(1-\varepsilon)D}, \quad (z \neq 0 \text{ and } f(0)=0);$$

(iii)
$$\Delta_1 \le \frac{g(y)-g(0)}{y} = G_0 \le c = \frac{\varepsilon+\beta\gamma(1-\varepsilon)}{\gamma} \in I_1, \quad (y \neq 0 \text{ and } g(0)=0),$$

where
$$I_1 \equiv \left[\Delta_1, {K_1}^* ab\right];$$

(iv)
$$\Delta_0 \le \frac{h(x)-h(0)}{x} = H_0 \le d = \frac{\beta\varepsilon(1-\varepsilon)(\beta\gamma-1)}{\gamma^2} \in I_0, \quad (x \neq 0 \text{ and } h(0)=0),$$

where
$$I_0 \equiv \left[\Delta_0, {K_0}^* \left[\frac{[(ab-c)c]}{a^2}\right]\right];$$

(v)
$$\phi(0) = f(0) = g(0) = h(0) = 0,$$

where ε, β, δ, γ and D(to be defined later) are all positive with $(1-\varepsilon) > 0$ and $\beta\gamma > 1$. Furthermore, I_0 and I_1 are subintervals of the Routh-Hurwitz interval.

Next we state three new theorems which are our main results.

Theorem 2.1. Suppose that conditions (i)-(v) are satisfied with $p(t,x,\dot{x},\ddot{x},\dddot{x}) \equiv 0$, then the trivial solution of the equation (1.1) is globally asymptotically stable.

Theorem 2.2. In addition to conditions (i)-(v) being satisfied, suppose that

(vi)
$$p(t,x,\dot{x},\ddot{x},\dddot{x}) \equiv p(t) \quad \text{and} \quad |p(t)| \le M,$$

for all $t \le 0$, then there exists a constant σ, $(0 < \sigma < \infty)$ depending only on the constants ε, β, γ, δ and D such that every solution of (1.1) satisfies

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) \le e^{-\frac{1}{2}\sigma t}\left\{A_1 + A_2\int_{t_0}^{t} |p(\tau)|\, e^{\frac{1}{2}\sigma\tau} d\tau\right\}^2 \tag{2.1}$$

for all t $\ge t_0$, where the constant $A_1 > 0$, depends on ε, β, γ, δ, D as well as on $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dddot{x}(t_0)$; and the constant $A_2 > 0$ depends on ε, β, γ, δ and D.

Theorem 2.3. Suppose the conditions of the Theorem 2.2 are satisfied with condition (vi) replaced with

(vii)
$$|p(t,x,\dot{x},\ddot{x},\dddot{x})| \le (|x|+|y|+|z|+|w|)\theta(t),$$

where $\theta(t)$ is a non negative and continuous function of t such that $\int_0^t \theta(s)ds \le M < \infty$ is satisfied with a positive constant M. Then, there exists a constant K_0 which depends on M, K_1, K_2 and t_0 such that every solution $x(t)$ of the equation (1.1) satisfies

$$|x(t)| \le K_0, \qquad |\dot{x}(t)| \le K_0, \qquad |\ddot{x}(t)| \le K_0, \qquad \left|\dddot{x}(t)\right| \le K_0$$

for sufficiently large t.

Remark: We wish to remark here that while the Theorem 2.1 is on the global asymptotic stability of the trivial solution, the Theorems 2.2 and 2.3 deal with the boundedness and ultimate boundedness of the solutions respectively.
Notations: Throughout this paper $K, K_0, K_1, \ldots K_{14}$ will denote finite positive constants whose magnitudes depend only on the functions g, h and p as well as constants ε, β, γ, δ and D but are independent of solutions of the equation (1.1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

3 Preliminary Results

We shall use as a tool to prove our main results a Lypunov function $V(x, y, z, w)$ defined by

$$\begin{aligned} 2V = {} & [\beta(1-\varepsilon)x + \gamma y + \delta z + w]^2 + [(1-\varepsilon)D - 1](\delta z + w)^2 \\ & + \beta D[\varepsilon + (1-\varepsilon)D - 1]y^2 + \gamma(D-1)z^2 + \varepsilon D w^2 + \beta^2\varepsilon(1-\varepsilon)x^2 \\ & + 2\gamma\delta[(1-\varepsilon)^2 D - 1]yz, \end{aligned} \tag{3.1}$$

where $0 < \varepsilon < 1 - \varepsilon < 1$, $\frac{\gamma\delta}{\beta} > 1 - \varepsilon$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\varepsilon)[\gamma\delta - \beta(1-\varepsilon)]}{\gamma\delta - \beta\varepsilon}$ with $D > \frac{1}{(1-\varepsilon)^2}$.

The following lemmas are needed in the proofs of Theorems 2.1, 2.2 and 2.3.

Lemma 3.1. Subject to the assumptions of Theorem 2.1 there exist positive constants $K_i = K_i(\varepsilon, \beta, \gamma, \delta, D), i = 1, 2$ such that

$$K_1(x^2 + y^2 + z^2 + w^2) \le V(x, y, z, w) \le K_2(x^2 + y^2 + z^2 + w^2). \tag{3.2}$$

Proof. First, it is clear from the equation (3.1) that $V(0,0,0,0) \equiv 0$. Next, noting that

$$\begin{aligned} \gamma(D-1)z^2 + 2\gamma\delta[(1-\varepsilon)^2 D - 1]yz = \gamma(D-1)\left\{z + \frac{\delta}{(D-1)}[(1-\varepsilon)^2 D - 1]y\right\}^2 - \\ \frac{\gamma\delta^2}{(D-1)}[(1-\varepsilon)^2 D - 1]^2 y^2 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} & \beta D\varepsilon y^2 + 2\gamma\delta[(1-\varepsilon)^2 D - 1]yz = \beta D\varepsilon\left\{y + \frac{\gamma\delta}{\beta D\varepsilon}[(1-\varepsilon)^2 D - 1]z\right\}^2 \\ & -\frac{(\gamma\delta)^2}{\beta D\varepsilon}\left\{[(1-\varepsilon)^2 D - 1]z\right\}^2, \end{aligned} \tag{3.4}$$

equations (3.3) and (3.4) can be utilized in such way to make it easier for equation (3.1) to be re-written as

$$
\begin{aligned}
2V = \{\beta(1-\varepsilon)x+\gamma y+\delta z+w\}^2+[(1-\varepsilon)D-1](\delta z+w)^2 \\
+\beta D\varepsilon\left\{y+\frac{\gamma\delta}{\beta D\varepsilon}[(1-\varepsilon)^2D-1]z\right\}^2 \\
+\beta^2\varepsilon(1-\varepsilon)x^2+\beta D[(1-\varepsilon)D-1]y^2 \\
+\gamma(D-1)z^2+\varepsilon Dw^2-\frac{(\gamma\delta)^2}{\beta D\varepsilon}\left\{[(1-\varepsilon)^2D-1]\right\}^2z^2.
\end{aligned}
\tag{3.5}
$$

Further rearrangement of the equation (3.5) gives

$$
\begin{aligned}
2V = \{\beta(1-\varepsilon)x+\gamma y+\delta z+w\}^2+[(1-\varepsilon)D-1](\delta z+w)^2 \\
+\beta D\varepsilon\left\{y+\frac{\gamma\delta}{\beta D\varepsilon}[(1-\varepsilon)^2D-1]z\right\}^2 \\
+\beta^2\varepsilon(1-\varepsilon)x^2+\beta D[(1-\varepsilon)D-1]y^2 \\
+\frac{\gamma\beta D\varepsilon(D-1)+2(\gamma\delta)^2D(1-\varepsilon)^2-(\gamma\delta)^2[(1-\varepsilon)^4+1]}{\beta D\varepsilon}z^2+\varepsilon Dw^2.
\end{aligned}
\tag{3.6}
$$

From the equation (3.6), we note that

$$
\begin{aligned}
2V \geq \beta^2\varepsilon(1-\varepsilon)x^2+\beta D[(1-\varepsilon)D-1]y^2 \\
+\frac{\gamma\beta\varepsilon(D-1)+2(\gamma\delta)^2D(1-\varepsilon)^2-(\gamma\delta)^2[(1-\varepsilon)^4+1]}{\beta D\varepsilon}z^2+\varepsilon Dw^2.
\end{aligned}
\tag{3.7}
$$

It follows from the above that there exists a constant K_1 such that

$$
V \geq K_1(x^2+y^2+z^2+w^2), \tag{3.8}
$$

where

$$
K_1 = \frac{\min}{2}\left\{\beta^2\varepsilon(1-\varepsilon), \beta D[(1-\varepsilon)D-1], \frac{\gamma\beta\varepsilon(D-1)+2(\gamma\delta)^2D(1-\varepsilon)^2-(\gamma\delta)^2[(1-\varepsilon)^4+1]}{\beta D\varepsilon}, \varepsilon D\right\}. \tag{3.9}
$$

On the other hand, let inequality $|y||z| \leq \frac{1}{2}(y^2+z^2)$ be used in equation (3.1) then,

$$
\begin{aligned}
&2V \leq \beta(1-\varepsilon)[\beta+\gamma+\delta+1]x^2+\left\{\gamma\left(\gamma+\beta(1-\varepsilon)+\delta[(1-\varepsilon)^2D-1]\right)\right. \\
&+\beta D[\varepsilon+(1-\varepsilon)D-1]y^2 \\
&+\left\{\delta^2(1-\varepsilon)D+\delta[\beta(1-\varepsilon)+(1-\varepsilon)D-1]+\gamma([1-\varepsilon]^2D-1)+(D-1)\right\}z^2 \\
&+\{\beta(1-\varepsilon)+(D-1)\delta+(1-\varepsilon)D\}w^2.
\end{aligned}
\tag{3.10}
$$

Consequently, it is not difficult to establish from the inequality (3.10) that

$$
V \leq K_2(x^2+y^2+z^2+w^2), \tag{3.11}
$$

where

$$
\begin{aligned}
&K_2 = \frac{\max}{2}\left\{\beta(1-\varepsilon)[\beta+\gamma+\delta+1];\gamma\left(\gamma+\beta(1-\varepsilon)+\delta[(1-\varepsilon)^2D-1]\right)+\beta D[\varepsilon+(1-\varepsilon)D-1];\right. \\
&\delta^2(1-\varepsilon)D+\delta[\beta(1-\varepsilon)+(1-\varepsilon)D-1]+\gamma([1-\varepsilon]^2D-1)+(D-1); \\
&\beta(1-\varepsilon)+(D-1)\delta+(1-\varepsilon)D.
\end{aligned}
\tag{3.12}
$$

At last, combination of inequalities (3.8) and (3.11) gives

$$
K_1(x^2+y^2+z^2+w^2) \leq V(x,y,x,w) \leq K_2(x^2+y^2+z^2+w^2). \tag{3.13}
$$

Lemma 3.2. In addition to assumptions of Theorem 2.1, let the condition (vi) of the Theorem 2.2 be satisfied also. Then there are positive constants $K_j = K_j(\varepsilon,\ \beta,\ \gamma,\ \delta, D)(j = 3,4)$ such that for any solution (x,y,z,w) of the system (1.2),

$$\dot{V}|_{(1.2)} \equiv \tfrac{d}{dt}V|_{(1.2)}(x,y,z,w) \leq -K_3(x^2+y^2+z^2+w^2)+K_4(|x|+|y|+|z|+|w|)\,|p(t)|. \qquad (3.14)$$

Proof. Differentiate (3.1) along the solution path of the system (2.1) to obtain

$$\begin{aligned}\dot{V} &= \{\beta(1-\varepsilon)x+\gamma y+\delta z+w\}\{\beta(1-\varepsilon)y+\gamma z+\delta w+\dot{w}\}\\ &+[(1-\varepsilon)D-1](\delta z+w)\{(\delta w+\dot{w})\}+\beta D[\varepsilon+(1-\varepsilon)D-1]yz\\ &+\gamma(D-1)zw+\varepsilon Dw\dot{w}+\beta^2\varepsilon(1-\varepsilon)xy+\gamma\delta[(1-\varepsilon)^2D-1]\left\{z^2+yw\right\}.\end{aligned} \qquad (3.15)$$

Using the system (1.2) in the equation (3.15) and on simplifying, we note that

$$\begin{aligned}\dot{V} &= \{\beta(1-\varepsilon)x+\gamma y+\delta z+w\}\{\beta(1-\varepsilon)y+\gamma z+\delta w\}\\ &+[(1-\varepsilon)D-1](\delta z+w)(\delta w)+\beta D[\varepsilon+(1-\varepsilon)D-1]yz\\ &+\gamma(D-1)zw+\beta^2\varepsilon(1-\varepsilon)xy+\gamma\delta[(1-\varepsilon)^2D-1]\left\{z^2+yw\right\}\\ &+[\beta(1-\varepsilon)x+\gamma y+\delta z+w]\left\{-\phi(w)-f(z)-g(y)-h(x)+p(t,x,\dot{x},\ddot{x},\dddot{x})\right\}\\ &+\{[[(1-\varepsilon)D-1](\delta z+w)+\varepsilon Dw]\}\left\{-\phi(w)-f(z)-g(y)-h(x)+p(t,x,\dot{x},\ddot{x},\dddot{x})\right\}.\end{aligned} \qquad (3.16)$$

Furthermore, a careful re-arrangement of the equation (3.16) gives

$$\begin{aligned}\dot{V} &= -w\phi(w)-\delta(1-\varepsilon)Dzf(z)-\gamma yg(y)-\beta(1-\varepsilon)xh(x)\\ &-\left[\beta(1-\varepsilon)\frac{g(y)}{y}+\gamma\frac{h(x)}{x}-\beta^2(1-\varepsilon)^2-\beta^2\varepsilon(1-\varepsilon)\right]xy-\\ &-\left[\beta(1-\varepsilon)\frac{f(z)}{z}+\delta(1-\varepsilon)D\frac{h(x)}{x}-\beta\gamma(1-\varepsilon)\right]xz\\ &-\left[\beta(1-\varepsilon)\frac{\phi(w)}{w}+\frac{h(x)}{x}-\beta\delta(1-\varepsilon)\right]xw-\\ &-\left[\gamma\frac{f(z)}{z}+\delta(1-\varepsilon)D\frac{g(y)}{y}-\gamma^2-\beta\delta(1-\varepsilon)-\beta D[\varepsilon+(1-\varepsilon)D-1]\right]yz\\ &-\left[\gamma\frac{\phi(w)}{w}+\frac{g(y)}{y}-\beta(1-\varepsilon)-\gamma\delta(1-\varepsilon)^2D\right]yw-\\ &-\left[\delta(1-\varepsilon)D\frac{\phi(w)}{w}+\frac{f(z)}{z}-\gamma D-(1-\varepsilon)D\delta^2\right]zw\\ &+\beta\gamma(1-\varepsilon)y^2+[\gamma\delta(1-\varepsilon)D]z^2+[\delta(1-\varepsilon)D]w^2\\ &+p\,[\beta(1-\varepsilon)x+\gamma y+(1-\varepsilon)D\delta z+w],\end{aligned} \qquad (3.17)$$

which becomes

$$\begin{aligned}\dot{V} &\leq -D\Phi_0 w^2-\delta(1-\varepsilon)F_0Dz^2-\gamma G_0y^2-\beta(1-\varepsilon)H_0x^2\\ &-\left\{\beta(1-\varepsilon)G_0+\gamma H_0-\beta^2(1-\varepsilon)^2-\beta^2\varepsilon(1-\varepsilon)\right\}xy\\ &-\{\beta(1-\varepsilon)F_0+\delta(1-\varepsilon)DH_0-\gamma\beta(1-\varepsilon)\}xz-\{\beta(1-\varepsilon)\Phi_0+H_0-\delta\beta(1-\varepsilon)\}xw\\ &-\left\{\gamma F_0+\delta(1-\varepsilon)DG_0-\gamma^2-\beta\delta(1-\varepsilon)-\beta D[D+(1-\varepsilon)D-1]\right\}yz\\ &-\left\{\gamma\Phi_0+G_0-\beta(1-\varepsilon)-\gamma\delta(1-\varepsilon)^2D\right\}yw\\ &-\left\{\delta(1-\varepsilon)D\Phi_0+F_0-\gamma D-\delta^2(1-\varepsilon)D\right\}zw+\beta\gamma(1-\varepsilon)y^2\\ &+[\gamma\delta(1-\varepsilon)D]z^2+[\delta(1-\varepsilon)D]w^2+p\,[\beta(1-\varepsilon)x+\gamma y+(1-\varepsilon)D\delta z+w]\end{aligned} \qquad (3.18)$$

on using the hypotheses on ϕ, f, g and h of the Theorem 2.1. The preceding inequality implies that

$$\dot{V} \leq -\varepsilon(w^2+z^2+y^2+x^2)+p\left[\beta(1-\varepsilon)x+\gamma y+(1-\varepsilon)D\delta z+w\right], \tag{3.19}$$

for some constants ε, γ, δ and D.

Hence,

$$\dot{V} \leq -\varepsilon(w^2+z^2+y^2+x^2)+K_3 p(|x|+|y|+|z|+|w|), \tag{3.20}$$

where

$$K_3 = \max\{\beta(1-\varepsilon),\gamma,(1-\varepsilon)D\delta,1\}. \tag{3.21}$$

To complete the proof of Lemma 3.2, choose $\varepsilon = K_4$ (K_4 a constant), then

$$\dot{V} \leq -K_4(w^2+z^2+y^2+x^2)+K_3 p(|x|+|y|+|z|+|w|). \tag{3.22}$$

4 Proof of Main Results

We shall now give the proofs of the main results.

Proof (Proof of Theorem 2.1). The proof of Theorem 2.1 follows from Lemmas 3.1 and 3.2 where it has been established that the trivial solution of the equation (1.1) is globally asymptotically stable. i.e every solution $(x(t),\dot{x}(t),\ddot{x}(t),\dddot{x}(t))$ of the system (1.2) satisfies $x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof (Proof of Theorem 2.2). Indeed, by using the inequality (3.22), it follows that

$$\frac{dV}{dt} \leq -K_4(x^2+y^2+z^2+w^2)+K_3(x^2+y^2+z^2+w^2)^{\frac{1}{2}}|p(t)|.$$

Again, it also follows from the inequality (3.8) that

$$(x^2+y^2+z^2+w^2)^{\frac{1}{2}} \leq \left(\frac{V}{K_1}\right)^{\frac{1}{2}}.$$

Thus, the inequality (3.22) becomes

$$\dot{V} \leq -K_5 V + K_6 V^{\frac{1}{2}}|p(t)|. \tag{4.1}$$

We note that $K_4(x^2+y^2+z^2+w^2) = K_4 \cdot \frac{V}{K_1}$ and

$$\frac{dV}{dt} \leq -K_5 V + K_6 V^{\frac{1}{2}}|p(t)|, \tag{4.2}$$

where $K_5 = \frac{K_4}{K_1}$ and $K_6 = \frac{K_3}{K_2^{\frac{1}{2}}}$.

Furthermore, the above inequality can be written as

$$\dot{V} \leq -2K_7 V + K_6 V^{\frac{1}{2}}|p(t)|, \tag{4.3}$$

where $K_7 = \frac{1}{2}K_5$.

Therefore

$$\dot{V} + K_7 V \leq -K_7 V + K_6 V^{\frac{1}{2}}|p(t)|. \tag{4.4}$$

Choosing a constant K_8 such that $K_8 = \frac{K_7}{K_6}$ gives

$$\dot{V} + K_7 V \leq K_6 V^{\frac{1}{2}} \left\{ |p(t)| - K_8 V^{\frac{1}{2}} \right\}. \tag{4.5}$$

Thus the inequality (4.5) becomes

$$\dot{V} + K_7 V \leq K_6 V^{\frac{1}{2}} V^* \tag{4.6}$$

where

$$V^* = |p(t)| - K_8 V^{\frac{1}{2}} \tag{4.7}$$

$$\leq V^{\frac{1}{2}} |p(t)|$$

$$\leq |p(t)|. \tag{4.8}$$

When $|p(t)| \leq K_8 V^{\frac{1}{2}}$,

$$V^* \leq 0, \tag{4.9}$$

and when $|p(t)| \geq K_8 V^{\frac{1}{2}}$,

$$V^* \leq |p(t)| \cdot \frac{1}{K_8}. \tag{4.10}$$

On substituting the inequality (4.9) into the inequality (4.5), we have

$$\dot{V} + K_7 V \leq K_9 V^{\frac{1}{2}} |p(t)|,$$

where

$$K_9 = \frac{K_6}{K_8}.$$

This implies that

$$V^{-\frac{1}{2}} \dot{V} + K_7 V^{\frac{1}{2}} \leq K_9 |p(t)|. \tag{4.11}$$

Multiplying both sides of the inequality (4.11) by $e^{\frac{1}{2}K_7 t}$, gives

$$e^{\frac{1}{2}K_7 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_7 V^{\frac{1}{2}} \right\} \leq e^{\frac{1}{2}K_7 t} K_9 |p(t)|, \tag{4.12}$$

i.e

$$2 \frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_7 t} \right\} \leq e^{\frac{1}{2}K_7 t} K_9 |p(t)|. \tag{4.13}$$

Integrating both sides of (4.13) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_7 \gamma} \right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_6 \tau} K_9 |p(\tau) d\tau|, \tag{4.14}$$

which implies that

$$\left\{ V^{\frac{1}{2}}(t) \right\} e^{\frac{1}{2}K_7 t} \leq V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_7 t_0} + \frac{1}{2} K_9 \int_{t0}^{t} |p(\tau)| e^{\frac{1}{2}K_7 \tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_7 t} \left\{ V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_7 t_0} + \frac{1}{2} K_9 \int_{t0}^{t} |p(\tau)| e^{\frac{1}{2}K_7 \tau} d\tau \right\}.$$

On utilizing inequalities (3.8) and (3.11), we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t)) \leq \\ e^{-\frac{1}{2}K_7 t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}^2(t_0)) \\ e^{\frac{1}{2}K_7 t_0}+\frac{1}{2}K_9\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_7\tau}d\tau\right\}^2, \tag{4.15}$$

for all $t \geq t_0$.
Thus,

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) \leq \\ \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_7 t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}^2(t_0))e^{\frac{1}{2}K_7 t_0} \\ +\frac{1}{2}K_9\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_7\tau}d\tau\right\}^2\right\}$$

$$\leq \left\{e^{-\frac{1}{2}K_7 t}\left\{A_1+A_2\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_7\tau}d\tau\right\}^2\right\}, \tag{4.16}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, \ldots K_9$ and $(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}^2(t_0))\}$.
By substituting $K_7 = \mu$ in the inequality (4.16), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) \leq \left\{e^{-\frac{1}{2}\mu t}\left\{A_1+A_2\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2\right\}.$$

This completes the proof.

Proof (Proof of Theorem 2.3). From the definition of function V and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can be obtained as

$$V \geq K_1\left(x^2+y^2+z^2+w^2\right), \tag{4.17}$$

and since $p \neq 0$ we can revise the conclusion of Lemma 3.2, i.e,

$$\dot{V} \leq -K_4(x^2+y^2+z^2+w^2)+K_3(|x|+|y|+|z|+|w|)\,|p(t)|,$$

and we obtain by using the condition (vii) as stated in the Theorem 2.3 that

$$\dot{V} \leq K_3(|x|+|y|+|z|+|w|)^2\theta(t). \tag{4.18}$$

By using the Schwartz inequalities on (4.18), we have

$$\dot{V} \leq K_{10}(x^2+y^2+z^2+u^2+w^2)\theta(t), \tag{4.19}$$

where $K_{10} = 2K_3$.
From inequalities (4.17) and (4.19) we have,

$$\dot{V} \leq K_{10}V\theta(t). \tag{4.20}$$

Integrating inequality (4.20) from 0 to t gives

$$V(t)-V(0) \leq K_{11}\int_0^t V(s)\theta(s)ds, \tag{4.21}$$

where $K_{11} = \frac{K_{10}}{K_1} = \frac{2K_3}{K_1}$. Thus,

$$V(t) \leq V(0) + K_{11} \int_0^t V(s)\theta(s)ds. \tag{4.22}$$

Applying the Grownwall-Reid-Bellman theorem on the inequality (4.22) yields

$$V(t) \leq V(0)exp\left(K_{11} \int_0^t \theta(s)ds\right). \tag{4.23}$$

This completes the proof of Theorem 2.3.

References

[1] A.M.A. Abou-El-Ela, A.I. Sadek, On the asymptotic behaviour of solutions of some differential equations of the fourth-order, *Ann. of Diff. Eqs.*, 8(1), 1-12(1992). MR1166444 (93c:34110).

[2] O.A. Adesina, On the qualitative behaviour of solutions of some-fourth order non-linear differential equations, *Ordinary Differential equations, Proc. Natl. Math. Cent. Abuja*, Vol.1 No.1, 87–93(2000), (Ezeilo, J.O.C., Tejumola, H.O. and Afuwape, A.U. Eds.), MR1991406.

[3] O.A. Adesina, B.S. Ogundare, New results on the convergence of solution for a certain fourth order nonlinear differential equation, *Journal of Computational Analysis and Applications* (Accepted for publication).

[4] A.U. Afuwape, On the convergence of solutions of certain fourth order differential equations, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.,*, 27, no. 1, 133–138(1981). MR0618718 (82j:34032).

[5] A.U. Afuwape, On some properties of solutions for certain fourth-order nonlinear differential equations, *Analysis*, 5, no. 1–2, 175-183(1985). MR0791498 (87a:34037).

[6] A.U. Afuwape, Convergence of the solutions for the equation $x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$, *Internat. J. Math. Math. Sci.*, 11, no. 4, 727–733(1988). MR0959453 (89i:34071).

[7] A.U. Afuwape, On the existence of a limiting regime in the sense of Demidovic for a certain fourth-order nonlinear differential equation, *J. Math. Anal. Appl.*, 129, no. 2 389–393(1988). MR0924297 (89a:34016).

[8] A.U. Afuwape, Uniform dissipativity of some fourth-order nonlinear differential equations, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.*, no 5, 35, 9–16(1989). MR1049162.

[9] A.U. Afuwape,Uniform dissipative solutions for some fourth-order nonlinear differential equations, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.*, no 4, 37, 431–445(1991). MR1249344 (94m:34087).

[10] P.S.M. Chin, Stability results for the solutions of certain fourth-order autonomous differential equations, *Internat. J. Control.*, 49, no. 4, 1163–1173(1989). MR0995162 (90d:34106)

[11] J.O.C. Ezeilo, New properties of the equation $\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$ for certain special values of the incrementary ratio $y^{-1}\{h(x+y) - h(x)\}$, *Equations differentielles et fonctionnelles non linires (Actes Conference Internat "Equa-Diff 73", Brussels/Louvain-la-Neuve)*, Hermann, Paris, 447–462(1973) . MR0430413 (55 #3418).

[12] B.S. Ogundare, Boundedness of solutions to fourth order differential equations with oscillatory restoring and forcing terms, *Electron. J. Differential Equations*, no 6, 6pp(2006). MR2198919 (2007m:34090).

[13] B.S. Ogundare, G.E. Okecha, Convergence of solutions of certain fourth order nonlinear differential equations, *Int. J. Math. Sci.*, Art. ID 12536, 13pp(2007). MR2320772 (2008b:34089).

[14] B.S. Ogundare, G.E. Okecha, Boundedness and stability properties of solution to certain fourth order nonlinear differential equation, *Nonlinear Stud.*, 15, no 1, 61–70(2008). MR2391313 (2009b:34161).

[15] R. Reissig, G. Sansone, R. Conti, *Non-linear differential equations of higher order*. Translated from German. Noordhoff International Publishing, Leyden, (1974). MR0344556 (49 #9295).

[16] A.I. Sadek, On the asymptotic behaviour of solutions of certain fourth-order ordinary differential equations, *Math. Japon.*, 45, no. 3, 527-540(1997). MR1456383 (98e:34069).

[17] A.I. Sadek, A.S. Al-Elaiw, Asymptotic behaviour of the solutions of a certain fourth-order differential equation, *Ann. Differential Equations*, 20, no. 3, 221-234(2004). MR2099957.

[18] H.O. Tejumola, Periodic boundary value problems for some fifth, fourth and third order ordinary differential equations, *J. Nigerian Math. Soc.*, 21, 37–46(2006). MR2376824.

[19] A. Tiryaki, C. Tunç, Construction Lyapunov functions for certain fourth-order autonomous differential equations, *Indian J. Pure Appl. Math*., 26, no. 3, 225-292(1995). MR1326483 (96a:34111).

[20] A. Tiryaki, C. Tunç, Boundedness and stability properties of solutions of certain fourth order differential equations via the intrinsic method, *Analysis*, 16, no. 4, 325–334(1996). MR1429457 (97m:34095).

[21] C. Tunç, A note on the stability and boundedness results of solutions of certain fourth order differential equations, *Appl. Math. Comput,* 155, no. 3, 837–843(2004). MR2078694 (2005e:34151).

[22] C. Tunç, Some stability and boundedness results for the solutions of certain fourth order differential equations, *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica,* 44, 161–171(2005). MR2218575 (2006k:34147).

[23] C. Tunç, Stability and boundedness of solutions to certain fourth-order differential equations, *Electron. J. Differential Equations*, No. 35, 10 pp (2006). MR2213579 (2007a:34085).

[24] C. Tunç, Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order, *Comput. Appl. Math*, 26, no. 1, 1-17 (2007). MR2320255 (2008c:34109).

[25] C. Tunç, About stability and boundedness of solutions of certain fourth order differential equations, *Nonlinear Phenom. Complex Syst.*, 9, no. 4, 380–387(2006). MR2352613 (2008f:34124).

[26] X. Wu, K. Xiong, Remarks on stability results for the solutions of certain fourth-order autonomous differential equations, *Internat. J. Control,* 69, no. 2, 353–360(1998). MR1684744 (2000d:34104).