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REMARKS ON THE STABILISATION OF STOCHASTIC DELAY DIFFERENTIAL EQUATIONS BY NOISE

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Abstract: This paper is concerned with the stabilisation of finite dimensional stochastic delay differential equation of the form

$$x'(t) = f_1\big(t, x(t), x(t-\tau_1(t), ..., x(t-\tau_n(t)))\big) + \int_{t-\tau_0(t)}^{t} f_2(s,t,x(s))ds, \qquad t \geq 0,$$

where $n \in \mathbf{N}$,$f_1 \in C(\mathbf{R}^+ \times (\mathbf{R}^d)^{n+1}; \mathbf{R}^d)$,$f_2 \in C(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^d; \mathbf{R}^d)$. Under more general conditions, we show that a linear multiplicative noise can always stabilise a general finite-dimensional functional differential equation whenever the delay is sufficiently small. Our results complement and improve existing results, and are also in line with the results of Appleby [1] which in itself generalised some well known results in the literature.

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noise

1. Introduction

Stochastic differential equations have received the attention of many authors, see for instance [1-6]. In an interesting paper [3], Mao proved that general finite dimensional stochastic differential equations can be stabilised or destabilised by Brownian motion. An equally interesting question arose from this: given an unstable functional differential equation (1.1), can it be stabilised by stochastic perturbation of the form $\sigma x(t)\dot{B}$? Here B(t) is a Brownian motion and σ is a real number. Appleby [1] answered this question affirmatively when he considered stochastic perturbations of the deterministic functional differential equation

$$x'(t) = f_1(t, x(t), x(-\tau_1(t)), .., x(t-\tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(s, t, x(s))ds, \quad t \geq 0 \quad (1.1)$$

where $n \in \mathbf{N}$ and functions $f_1 \in C(\mathbf{R}^+ \times (\mathbf{R}^d)^{n+1}; \mathbf{R}^d)$, $f_2 \in C(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^d; \mathbf{R}^d)$. Moreover it was assumed that f_1 and f_2 satisfy the following global Lipschitz conditions in the space parameters x , so that for all $t \geq 0$, there exists $L > 0$ such that

$$\| f_1(t, x_1, ..., x_{n+1}) - f_1(t, y_1, ..., y_{n+1}) \| \leq L \sum_{j=1}^{n+1} \| x_j - y_j \|, \; x_j, y_j \in \mathbf{R}^d, \quad (1.2)$$

$$\| f_2(s, t, x) - f_2(s, t, y) \| \leq L \| x - y \|, \qquad 0 \leq s \leq t, \qquad x, y \in \mathbf{R}^d. \quad (1.3)$$

Furthermore, Appleby [1] showed that if both conditions (1.2) and (1.3) are satisfied and, moreover, if the time delays τ_i, $0 \leq i \leq n$ are all sufficiently small, then it is possible to choose σ (a real constant) sufficiently large so that the equation

$$dX(t) = \Big(f_1\big(t, X(t), X(t-\tau_1(t)), X(t-\tau_2(t)), ..., X(t-\tau_n(t))\big) + \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))ds \Big) dt + \sigma X(t)dB(t), \qquad t \geq 0 \quad (1.4)$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0, \quad (1.5)$$

will be almost surely exponentially stable.

In this paper, we seek to use the approach of Appleby [1] to weaken Lipschitz conditions (1.2) and (1.3) so that

$$\| f_1(t, x_1, ..., x_{n+1}) - f_1(t, y_1, ..., y_{n+1}) \| \leq P(t) \sum_{j=1}^{n+1} \| x_j - y_j \|,$$

$$x_j, y_j \in \mathbf{R}^d, \quad (1.6)$$

$$\| f_2(s, t, x) - f_2(s, t, y) \| \leq P(t) \| x - y \|, \qquad 0 \leq s \leq t, \qquad x, y \in \mathbf{R}^d, \quad (1.7)$$

where P(t) is a non negative process. Our results complement those of Appleby [1].

2. Preliminaries

Our approach in this work shall follow the work of Appleby [1]. Throughout this paper, unless otherwise specified, we use the following notation. Let $C(I; J)$ be the space of continuous functions taking the fin

$$\| f_1(t, x_1, x_2, ..., x_{n+1}) \| \leq \sum_{j=1}^{n+1} K_j \| x_j \| \qquad \forall \qquad t > 0 \qquad x_j \in \mathbf{R}^d, \quad (2.1)$$

$$\| f_2(s, t, x) \| \leq K_0 \| x \| \qquad \forall \qquad 0 \leq s \leq t, \qquad x \in \mathbf{R}^d, \quad (2.2)$$

where $k_j, = 0, ..., n+1$ are non-negative constants.

In what follows, we now place some assumptions on τ_j , $j = 0, 1, ..., n$. $\tau_j \in C(\mathbf{R}^+; \mathbf{R}^+)$ and $\bar{\tau}(t) = \max_{j=0,1,...,n} \tau_j(t)$ is a continuous function on $\mathbf{R}^+$. Furthermore, we suppose that $\sup_{t \geq 0} \bar{\tau}(t) = \bar{\tau}$, i.e, there exists a finite $\bar{\tau}$ satisfying

$$\bar{\tau} = \sup_{t \geq 0} \max_{j=0,1,...,n} \tau_j(t). \quad (2.3)$$

Let $\phi \in C([-\tau, 0], \mathbf{R}_d)$, where $\bar{\tau}$ satisfies (2.3). Now let

$$x(t) = \psi(t), \qquad -\bar{\tau} \leq t \leq 0, \quad (2.4)$$

then, according to Appleby [1], if f_1, f_2 satisfy $(1.2), (1.3), (2.1), (2.2)$ and τ_j satisfies the above continuity hypotheses, it follows that there is a unique solution of the equation (1.1). Note moreover that if $\psi(t) = 0, \forall - \bar{\tau} \leq t \leq 0$, then $x \equiv 0$ is the unique solution of the equation (1.1), which we call the zero solution of the equation (1.1). We refer the reader to Appleby [1] where the following almost sure asymptotic stability of solution to the equation (1.1) was established. Let $(B(t))_{t\geq 0}$ be a one dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$ with natural filtration $(\mathbf{F})_{t\geq 0}$, where $\mathbf{F}_t = \sigma(B(s) : 0 \leq s \leq t)$. Now, let σ be real constant and consider the stochastic functional differential equation.

$$dX(t) = \Big(f_1\big(t, X(t), X(t-\tau_1(t)), X(t-\tau_2(t)), ..., X(t-\tau_n(t))\big)$$

$$+ \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))ds \Big) dt + \sigma X(t) dB(t), \qquad t \geq 0 \tag{2.5}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0. \tag{2.6}$$

If the above assumptions on ψ, f_1, f_2 and $\tau_j, j = 0, ..., n$ hold, then equation (2.5) has a unique, strong continuous solution on $\mathbf{R}^+$. Again, if $\psi \equiv 0$, then $X(t) = 0$, $\forall t \geq 0$, almost surely. Observe that when $\sigma = 0$, the problem reduces to the deterministic functional differential equation (1.1). Hence the question of whether these are values of $\sigma = 0$ such that

$$\lim_{t\to\infty} \sup \frac{1}{t} log \parallel X(t) \parallel \leq -\lambda \quad a.s \tag{2.7}$$

was answered in [1].

3. Main Result

Suppose that assumptions (1.6) and (1.7) are satisfied. Suppose further that

$$\sigma^2 > 2 \sum_{2n+1}^{j=2} K_j$$

and let $\tau(\sigma^2, K)$ be given by

$$\tau(\sigma^2, K) = \frac{2}{\sigma^2} log \left(\frac{K_0 + \sqrt{K_0^2 + \sigma^2(\frac{1}{2}\sigma^2 - K_1)(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)}}{(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)} \right).$$

If $\bar{\tau}$ defined by

$$\bar{\tau} = \sup_{t\geq 0} \max_{j=0,...,n} \tau_j(t)$$

satisfies $\bar{\tau} < \tau(\sigma^2, K)$, then the solution of the equation

$$dX(t) = \Big(f_1\big(t, X(t), X(t-\tau_1(t)), X(t-\tau_2(t)), ..., X(t-\tau_n(t))\big)$$

$$+ \int_{t-\tau_0(t)}^{t} f_2(s, t, X(s))ds\Big)dt + \sigma X(t)dB(t), \qquad t \geq 0, \tag{3.1}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0, \tag{3.2}$$

satisfies

$$\lim_{t\to\infty} \sup \frac{1}{t} \log \| X(t) \| \leq -\lambda_1(\sigma^2, \bar{\tau}) < 0, \qquad a.s., \tag{3.3}$$

where λ_1 is given by

$$\lambda_1(\sigma^2, \bar{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2\bar{\tau}/2} - K_0 \frac{1}{\frac{\sigma^2}{2}}\big(e^{\sigma^2\bar{\tau}} - e^{\sigma^2\bar{\tau}/2}\big).$$

4. Proof of the Main Result

We shall need the following auxiliary functions in the proof of our main result. Suppose $\Phi(t) = I_d$, for $t \in [-\overline{\tau}, 0]$ and consider the matrix stochastic differential equation $d\Phi(t) = \sigma\Phi(t)dB(t)$. This has solution $\Phi(t) = \varphi(t)I_d$, where $(\varphi(t))_{t>0}$ is the scalar geometric Brownian motion which satisfies $d\varphi(t) = \sigma\varphi(t)dB(t)$ with $\varphi(t) = 1, t \in [\overline{\tau}, 0]$ and hence is given by

$$\begin{aligned} \varphi(t) &= e^{\int -\frac{\sigma^2}{2}dt + \int \sigma dB(t)} \qquad \text{(by Ito's formula)} \\ &= e^{\int -\frac{\sigma^2}{2}dt + \sigma B(t)}. \end{aligned} \tag{4.1}$$

Thus, $\Phi(t)^{-1}$ exists for all $t \geq -\overline{\tau}$ a.s., and in particular $\Phi(t)^{-1} = \varphi(t)^{-1}I_d$, so $\| \Phi(t) \|= \varphi(t)$, $\| \Phi(t)^{-1} \|= \varphi(t)^{-1}$. Now for $t \geq 0$, let

$$R(t) = f_1(t, X(t), ..., X(t-\tau_n(t))) + \int_{t-\tau_0(t)}^{t} f(s, t, x(s))ds$$

therefore,

$$dX = R(t)dt + \sigma X(t)dB(t).$$

On employing Ito's product rule, let

$$dX_1 = \sigma X_1(t)dB(t),$$

$$dX_2 = A(t)d(t) + C(t)dB(t),$$

$$\begin{aligned} dX &= [\sigma X_1(t)dB(t)]X_2(t) + [A(t)d(t) + C(t)dB(t)]X_1(t) + \sigma X_1(t)d(t) \\ &= \sigma(X_1(t)X_2(t))dB(t) + A(t)X_1dt + C(t)X_1(t)dB(t) + \sigma C(t)X_1(t)dt \\ &= \sigma(X(t))dB(t) + A(t)X_1dt + C(t)X_1(t)dB(t) + \sigma C(t)X_1(t)dt, \end{aligned}$$

then,

$$R(t)dt = A(t)X_1dt + \sigma C(t)X_1(t)dt + C(t)X_1(t)dB(t).$$

On choosing $C = 0$ and $A = R(t)X_1^{-1}$, we have

$$dX_1 = \sigma X_1(t)dB(t),$$

$$dX_2 = R(t)X_1^{-1}dt.$$

On employing Ito's lemma (see e.g. [1]), it follows that

$$X_1 = \exp(\int_0^t -\frac{\sigma^2}{2}ds + \int_0^t \sigma dB(t))$$

$$= e^{-\frac{\sigma^2}{2}+\sigma B(t)},$$

$$X_2 = \psi(0) + \int_0^t R(s)X_1^{-1}ds$$

$$X = X_1(\psi_0 + \int_0^t R(s)X_1^{-1}ds).$$

But $X_1 = \Phi(t)$, hence

$$X(t) = \Phi(t)(\psi(0) + \int_0^t \Phi(s)^{-1}R(s)ds), t \geq 0. \tag{4.2}$$

Now, let $y(t) = \Phi(t)^{-1}X(t)$ for $t \geq -\overline{\tau}$. As $t \longmapsto \Phi(t), R(t)$ is continuous, equation (4.2) implies $y \in C'(\mathbf{R}^+; \mathbf{R}^d)$. Thus for $t \geq 0$, $t - \tau_j(t) \geq -\overline{\tau}$, we have

$$\begin{aligned} y'(t) = \Phi(t)\big(f_1(t, \Phi(t)y(t), ..., \Phi(t-\tau_n(t)y(t-\tau_n(t))) \\ + \int_{t-\tau_0(t)}^t f_2(s, t, \Phi(s)y(s))ds\big). \end{aligned} \tag{4.3}$$

Thus,

$$\| y'(t) \| \leq \| K_1(t) \| \| y(t) \| + \sum_{j=2}^{n+1} \varphi(t)^{-1}\varphi(t-\tau_{j-1}(t)) \| K_j(t) \| \| y(t-\tau_{j-1}(t)) \|$$

$$+\int_{t-\tau_0(t)}^{t} \varphi(t)^{-1}\varphi(s) \parallel K_0(t) \parallel\parallel y(s) \parallel ds, \qquad (4.4)$$

and if we define $p(t)$ for $t \geq \overline{\tau}$ by

$$p(t) =\parallel K_1(t) \parallel + \sum_{j=2}^{n+1} \varphi(t)^{-1}\varphi(t-\tau_{j-1}(t)) \parallel K_j(t) \parallel$$
$$+\int_{t-\tau_0(t)}^{t} \varphi(t)^{-1}\varphi(s) \parallel K_0(t) \parallel\parallel y(s) \parallel ds, \qquad (4.5)$$

we have that for $t \geq \overline{\tau}$,

$$\parallel y(t) \parallel \geq \parallel y(\tau) \parallel + \int_{\overline{\tau}}^{t} p(s) \max_{s-\overline{\tau}\leq u\leq s} \parallel y(u) \parallel ds. \qquad (4.6)$$

On letting

$$y^*(t) = \max_{t-\overline{\tau}\leq s\leq t} \parallel y(s) \parallel, \qquad for \qquad t \geq \overline{\tau},$$

we have

$$y^*(t) \leq \parallel y(\overline{\tau}) \parallel + \int_{\overline{\tau}}^{t} p(s)y^*(s)ds.$$

On using Grownwall's inequality, it follows that

$$\parallel y(t) \parallel \leq y^*(t) \leq \parallel y(\overline{\tau}) \parallel e^{\int_{\overline{\tau}}^{t} p(s)ds}, \qquad t \geq \overline{\tau}.$$

If there exists a finite deterministic constant C, such that

$$\lim_{t\to\infty} \sup \frac{1}{t} \int_{2\overline{\tau}}^{t} p(s)ds \leq C \qquad (4.7)$$

$$\lim_{t\to\infty} \frac{1}{t} log \parallel \phi(t) \parallel = -\frac{\sigma^2}{2}, \qquad a.s.,$$

then the definition of y gives

$$\lim_{t\to\infty} \sup \frac{1}{t} log \parallel X(t) \parallel = -\frac{\sigma^2}{2} + C, \qquad a.s. \qquad (4.8)$$

We shall need the following results in the proof of our main results.

Lemma 1. *Suppose* $\tau \in C(R^+; [0, \overline{\tau}])$, *and* $\varphi(t)$ *satisfies*

$$\varphi(t) = e^{\int -\frac{\sigma^2}{2}dt+\sigma B(t)}, \qquad (4.9)$$

then,

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{2\overline{\tau}}^{t} \parallel K_j \parallel \varphi(s)^{-1}\varphi(s-\tau(s))ds \le K_j e^{\sigma^2\overline{\tau}}, \qquad a.s. \tag{4.10}$$

Lemma 2. *Let* $\tau \in C(\mathbf{R}^+;[0,\overline{\tau}])$ *and*

$$q_1(t) = \int_{t-\overline{\tau}(t)}^{t} \varphi(t)^{-1}\varphi(s)ds,$$

where $\varphi(t)$ *satisfies equation* (4.9)*. Then,*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{2\overline{\tau}}^{t} q_1(s)ds \le K_0 e^{\sigma^2\overline{\tau}/2}\int_0^{\overline{\tau}} e^{\sigma^2 u/2}du, \qquad a.s. \tag{4.11}$$

Proof Lemma 1. From the equation (4.9),

$$\varphi(s)^{-1}\varphi(s-\tau(s)) \le e^{\sigma^2\overline{\tau}/2}e^{-\sigma(B(s)-B(s-\tau(s)))}. \tag{4.12}$$

For each fixed $t \ge \overline{\tau}$, as $0 \le \tau(t) \le \overline{\tau}$, a.s., we have

$$B(t) - B(t-\tau(t)) \le \max_{t-\overline{\tau}\le s\le t} B(t) - B(s) = \max_{0\le s\le \overline{\tau}} W_s(t),$$

where $s \longmapsto W_s(t)$ is a standard Brownian motion. As $\max_{0\le s\le\overline{\tau}} W_s(t)$ has the same distribution as $|W_{\overline{\tau}}(t)|$ and

$$E[e^{\lambda|X|}] = e^{\frac{1}{2}\lambda^2\alpha^2}, \qquad for X\sim\mathcal{N}(0,\alpha^2),$$

we get,

$$E[e^{B(t)-B(t-\tau(t))}] \le E[e^{\max_{0\le s\le\overline{\tau}} W_s(t)}] = E[e^{|W_{\overline{\tau}}(t)|}] = e^{\frac{1}{2}\overline{\tau}}$$

$$E[e^{4(B(t)-B(t-\tau(t)))}] \le e^{8\overline{\tau}}. \tag{4.13}$$

Since $K_j(t)\sim\mathcal{N}(K_j, D_j), j = 1,...,n$ are independent random variables,

$$E[\parallel K_j(t) \parallel e^{\sigma|W_{\overline{\tau}}(t)|}] = E[\parallel K_j(t) \parallel]E[e^{\sigma|W_{\overline{\tau}}(t)|}] \le \parallel E[K_j(t)] \parallel e^{\frac{\sigma}{2}\overline{\tau}}$$

$$= \parallel K_j \parallel e^{\frac{\sigma}{2}\overline{\tau}} = |K_j|e^{\frac{\sigma}{2}\overline{\tau}}, \qquad j = 1,...,n. \tag{4.14}$$

Since K_j are constants,

$$E[\parallel K_j(t) \parallel e^{4\sigma(B(t)-B(t-\tau(t)))}] \le |K_j|e^{8\sigma\overline{\tau}}. \tag{4.15}$$

Define for $n \in \mathbf{N}$, the two sequence of random variables

$$X_n = \int_{(2n+1)\overline{\tau}}^{(2n+2)\overline{\tau}} \parallel K_j(t) \parallel e^{\sigma(B(s)-B(t-\tau(s)))} ds \qquad j = 1, ..., n,$$

$$Y_n = \int_{(2n)\overline{\tau}}^{(2n+1)\overline{\tau}} \parallel K_j(t) \parallel e^{\sigma(B(s)-B(t-\tau(s)))} ds \qquad j = 1, ..., n.$$

By Holders inequality and (4.19), we get

$$E[X_n] \leq |K_j|\overline{\tau}e^{\frac{\sigma}{2}\overline{\tau}},$$

$$E[Y_n] \leq |K_j|\overline{\tau}e^{\frac{\sigma}{2}\overline{\tau}},$$

$$E[X_n^4] \leq |K_j|\overline{\tau}e^{8\sigma\overline{\tau}},$$

$$E[Y_n^4] \leq |K_j|\overline{\tau}e^{8\sigma\overline{\tau}}.$$

Finally, as $0 \leq \tau(t) \leq \overline{\tau}$, we see that $\{X_n\}$ is a sequence of independent random variables, as is $\{Y_n\}$. By, Strong Law of Large Numbers (see [8]),

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=1}^{n} X_j \leq |K_j|\overline{\tau}e^{\frac{\sigma}{2}\overline{\tau}}, \tag{4.16}$$

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=1}^{n} Y_j \leq |K_j|\overline{\tau}e^{\frac{\sigma}{2}\overline{\tau}}. \tag{4.17}$$

With $t \geq 4\overline{\tau}$, there is $n = n(t) \in \mathbf{N}$ such that $(2n+2)\overline{\tau} \leq t < (2n+4)\overline{\tau}$, and

$$\frac{1}{t}\int_{t}^{(2)\overline{\tau}} \parallel K_j(t) \parallel e^{\sigma(B(s)-B(t-\tau(s)))} ds = \frac{n}{t}\Big(\frac{1}{n}\sum_{j=1}^{n} X_j + \frac{1}{n}\sum_{j=1}^{n} Y_j\Big)$$

$$+\frac{1}{t}\int_{(2n+2)\overline{\tau}}^{t} \parallel K_j(t) \parallel e^{\sigma(B(s)-B(t-\tau(s)))} ds \qquad j = 1, ..., n. \tag{4.18}$$

Since $\lim_{n\to\infty} \frac{n(t)}{t} = \frac{1}{2\overline{\tau}}$, we will show that

$$\lim_{t\to\infty} \frac{1}{t}\int_{(2n(t)+2)\overline{\tau}}^{t} \parallel K_j(t) \parallel e^{\sigma(B(s)-B(t-\tau(s)))} ds = 0. \tag{4.19}$$

By Markov's inequality and Borel Cantelli lemma,

$$E[U_n(4)] \leq C(n+1)^{-1} \qquad \text{for some} \qquad C > 0,$$

where $U_n = \frac{X_n+Y_n}{n+1}$. Thus,

$$\lim_{n\to\infty} U_n = 0.$$

This implies that (4.19) is true.

On using inequality (4.12), we have

$$\lim_{t\to\infty} \sup \frac{1}{t}\int_{2\overline{\tau}}^{t} \parallel K_j \parallel \varphi(s)^{-1}\varphi(s-\tau(s))ds \leq K_j e^{\sigma^2\overline{\tau}}, \qquad a.s.$$

Proof Lemma 2. From equation (4.9),

$$\varphi(t) = e^{\int -\frac{\sigma^2}{2}dt+\sigma B(t)},$$

and thus

$$\varphi(s)^{-1}\varphi(s-\tau(s)) \leq e^{\sigma^2\overline{\tau}/2}e^{-\sigma(B(s)-B(s-\tau(s)))}. \tag{4.20}$$

For each fixed $t-\overline{\tau}$, as $0 \leq \tau(t) \leq \overline{\tau}$, a.s, we have

$$B(t) - B(t-\tau(t)) \leq \max_{t-\overline{\tau}\leq s\leq t} B(t) - B(s) = \max_{0\leq s\leq \overline{\tau}} W_s(t),$$

where $s \mapsto W_s(t)$ is a standard Brownian motion (see [7]). As $\max_{0\leq s\leq\overline{\tau}}$ has the same distribution as $|W_{\overline{\tau}}(t)|$, and $E[e^{\lambda|x|}] = 2e^{\frac{1}{2}\lambda^2\alpha^2}$ for $X \sim N(0,\alpha^2)$,

$$E[e^{B(t)-B(t-\tau(t))}] \leq E[e^{max_{0\leq s\leq\overline{\tau}}W_s(t)}] = E[e^{|W_{\overline{\tau}}(t)|}] = e^{\frac{1}{2}\overline{\tau}} \tag{4.21}$$

$$E[e^{4(B(t)-B(t-\tau(t)))}] = e^{8\overline{\tau}}, \tag{4.22}$$

therefore,

$$E[K_0(t)e^{B(t)-B(t-\tau(t))}] = E[K_0(t)]E[B(t) - B(t-\tau(t))].$$

Since $K_0(t)$ is independent of $B(t) - B(t-\tau(t))$, then,

$$E[K_0(t)e^{B(t)-B(t-\tau(t))}] = K_0 e^{\frac{1}{2}\overline{\tau}}$$

$$E[e^{4(B(t)-B(t-\tau(t)))}] = K_0 e^{8\overline{\tau}}.$$

Set $q(t) = \int_{t-\overline{\tau}}^{t} K_0(t)e^{-\sigma(B(t)-B(s))}ds$. From (4.20),

$$K_0(t)\phi^{-1}\phi(s-\tau(s)) \leq e^{\sigma^2\frac{\overline{\tau}}{2}}K_0(t)e^{-\sigma(B(s)-B(s-\tau(s)))}. \tag{4.23}$$

Since $q(t) = \int_{t-\tau(t)}^{t} K_0(t)\phi^{-1}(t)\phi(s)ds$, then $q(t) \leq e^{\sigma\frac{\overline{\tau}}{2}}q(t)$. Thus we would be done if we can prove that

$$\lim_{t\to\infty} \frac{1}{t}\int_{2\overline{\tau}}^{t} q(s)ds = K_0\int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u}du \qquad a.s. \tag{4.24}$$

Now, let

$$X_n = \int_{(2n+1)\overline{\tau}}^{(2n+2)\overline{\tau}} q(s)ds$$

and

$$Y_n = \int_{(2n+1)\overline{\tau}}^{(2n+2)\overline{\tau}} q(s)ds \qquad n \geq 1,$$

where X_n and Y_n are sequences of independent random variables. Then,

$$E(X_n) = K_0\overline{\tau} \int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u} du$$

and

$$E(Y_n) = K_0\overline{\tau} \int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u} du.$$

Then by Strong Law of Large Numbers (see [8]),

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=1}^{n} X_j \leq K_0\overline{\tau} \int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u} du, \tag{4.25}$$

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=1}^{n} Y_j \leq K_0\overline{\tau} \int_0^{\overline{\tau}} e^{\frac{1}{2}\sigma^2 u} du, \tag{4.26}$$

$$\frac{1}{t}\int_{2\overline{\tau}}^{t} q(s)ds = \frac{n}{t}\left(\frac{1}{n}\sum_{j=1}^{n} X_j + \frac{1}{n}\sum_{j=1}^{n} Y_j\right) + \frac{1}{t}\int_{(2n+2)\overline{\tau}}^{t} q(s)ds. \tag{4.27}$$

Since $\lim_{t\to\infty} \frac{n(t)}{t} = \frac{1}{2\overline{\tau}}$, if we can show that

$$\lim_{t\to\infty} \frac{1}{t}\int_{(2n(t)+2)\overline{\tau}}^{t} q(s)ds = 0 \qquad a.s, \tag{4.28}$$

then, (4.24) is true. Let us observe that since $\lim_{n\to\infty U_n=0}$ a.s., where $U_n = \frac{X_n+Y_n}{n+1}$, then equation (4.28) holds. But this is true by Markov's inequality and the Borel Cantelli Lemma as

$$E[U_n^4] \leq C(n+1)^{-4},$$

for some $C > 0$. Thus the completion of the proof.

Let us now apply Lemma 1 to each of the terms in the sum on the right hand side of (4.5), and also applying Lemma 2 to the integral term yields equation

(4.5) with $C = K_1 + \sum_{j=2}^{n+1} K_j e^{\sigma^2 \overline{\tau}/2} + K_0 e^{\sigma^2 \overline{\tau}/2} \int_{\overline{\tau}}^{0} e^{\sigma^2 \overline{u}/2} du$. Now, for $\overline{\tau} > 0$, define

$$\lambda_1(\sigma^2, \overline{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2 \overline{\tau}/2} - K_0 \frac{1}{\sigma^2/2}\left(e^{\sigma^2 \overline{\tau}} - e^{\sigma^2 \overline{\tau}/2}\right).$$

Define also $K = (K_0, K_1, ..., K_{n+1})^T$.Then there exists a unique $\tau = \tau(\sigma^2, K) > 0$ such that $\lambda_1(\sigma^2, \tau(\sigma^2, K)) = 0$ and $\lambda_1(\sigma^2, \tau) > 0$ for all $\tau < \tau(\sigma^2, K)$. In fact, we can explicitly compute

$$\tau(\sigma^2, K) = \frac{2}{\sigma^2} log \left(\frac{K_0 + \sqrt{K_0^2 + \sigma^2(\frac{1}{2}\sigma^2 - K_1)(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)}}{(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)} \right)$$

which agrees with Appleby [1].

References

[1] J. Appleby, Stabilisation of functional differential equations by noise, *Systems Control Lett.*, To Appear.

[2] J. Appleby, X. Mao, Stochastic stabilisation of functional differential equations, *Systems Control Lett.*, **54** (2005), 1069-1081.

[3] X. Mao, Stochastic stabilization and destabilization, *Systems and Control Lett.*, **23** (1994), 279-290.

[4] X. Mao, Exponential stability in mean-square for stochastic differential equations, *Stochastic Analysis and its Applications*, **8**, No. 1 (1990), 91-103.

[5] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Volume 182 of Pure and Applied Mathematics, Marcel Dekker, New York (1994).

[6] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, New York (1997).

[7] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Volume 113 of Graduate Texts in Mathematics, Springer, New York (1991).

[8] D. Williams, *Probability with Martingales*, Cambridge University Press (1992).