Boundedness results for a certain third order nonlinear differential equation

Timothy A. Ademola a, Michael O. Ogundiran b, Peter O. Arawomo c, Olufemi Adeyinka Adesina d,*

a Department of Mathematics and Statistics, Bowen University, P.M.B. 284, Iwo, Osun State, Nigeria
b College of Natural and Applied Sciences, Department of Physical Sciences, Bells University of Technology, P.M.B 1015, Ota, Ogun State, Nigeria
c Department of Mathematics, University of Ibadan, Ibadan, Nigeria
d Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria

Keywords:
Bounded solutions
Ultimate bounded solution
Lyapunov functions
Third order differential equations

Abstract

Sufficient conditions for the existence of solutions to boundedness and ultimate boundedness problems associated to a certain third order nonlinear differential equation are given by means of the Lyapunov’s second method. The appropriate Lyapunov function is given explicitly. Our results complement some well known results on the third order differential equations in the literature.

1. Introduction

The question addressed in this paper is related to the study of boundedness and ultimate boundedness of solutions which is very important in the theory and applications of nonlinear differential equations. In the actual literature, many works have been done on these properties of solutions; see for instance Reissig et al. [16], Rouche et al. [17] and Yoshizawa [24] which contain general theorems on the subject matter. Notable authors that have contributed to the qualitative properties of solutions of nonlinear third order differential equations include Ademola et al. [1] on uniform asymptotic stability of solutions; Afuwape [2,3] and Hara [14] on ultimate boundedness of solutions; Afuwape and Adesina [5], Andres [6], Bereketoglu, and Győri [7], Ezeilo [8–12], Ezeilo and Tejumola [13], Swick [19], Tejumola [20] and Tunç [23] worked on boundedness of solutions. For the case when the considered third order equations are non-autonomous, we can mention the works of Qian [15], Swick [18] and Tunç [22] on asymptotic behaviour of solutions. Furthermore, Afuwape [4] and Tejumola [21] worked on periodic solutions.

Most of these works were done with the aid of Lyapunov functions. Unfortunately, with respect to our observation, these Lyapunov functions are either incomplete or contain signum functions. These we find too weak. Thus the purpose of this paper is to construct a complete Lyapunov function and use it to study boundedness (when $p = p(t, x, \dot{x}, \ddot{x})$ in (1.1)) and ultimate boundedness of solutions of the third order nonlinear differential equation

$$x + f(\dot{x}) + g(x) + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

or its equivalent system of differential equations

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= p(t, x, y, z) - f(z) - g(y) - h(x),
\end{align*}$$

(1.1)

or (1.2)

© 2010 Elsevier Inc. All rights reserved.
where \( f, g, h \in C(\mathbb{R}, \mathbb{R}) \), \( p \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \). \( \mathbb{R}^+ = [0, \infty) \) and \( \mathbb{R} = (-\infty, \infty) \). It is assumed that the functions \( f, g, h \) and \( p \) depend only on the arguments displayed explicitly, and the dots, as usual, denote differentiation with respect to the independent variable \( t \). We shall require that the derivative \( h'(x) = \frac{dh(x)}{dx} \) exists and continuous, also the uniqueness of \((1.1)\) or \((1.2)\) will also be assumed. The results obtained in this work improve, generalize and complement existing results on third order nonlinear differential equations in the literature.

2. Preliminaries

Our notations shall follow those of Afuwape [3] and Hara [14]. Consider the system of the form

\[
X' = F(t, X),
\]

where \( X \in \mathbb{R}^n, F : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function, \( \mathbb{R}^+ = [0, \infty) \) and \( \mathbb{R}^n \) is then Euclidean \( n \)-space.

**Definition 2.1.** The solutions of \((2.1)\) are uniformly ultimately bounded for bound \( B \), if there exists a \( B > 0 \) and if corresponding to any \( \varepsilon > 0 \), there exists a \( T(\varepsilon) > 0 \) such that whenever \( \|X_0\| = \|X(t, t_0, X_0)\| < \varepsilon \) then

\[
\|X(t, t_0, X_0)\| < B \quad \text{for all } t_0 \geq 0 \quad \text{and} \quad t \geq t_0 + T(\varepsilon).
\]

We now give a lemma which will play a major role in the proof of our results.

**Lemma 2.2.** Suppose that there exists a Lyapunov function \( V(t, X(t)) \) defined on \( \mathbb{R}^+, \|X(t)\| \geq K \) where \( K \) may be large, which satisfies the following conditions:

(i) \( a(\|X(t)\|) \leq V(t, X(t)) \leq b(\|X(t)\|) \), where \( a(r), b(r) \) are continuous and increasing and \( a(r) \rightarrow \infty \) as \( r \rightarrow \infty \);

(ii) \( V(t, X(t)) = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left[ V(t + h, X(t) + F(t, X(t))) - V(t, X(t)) \right] \leq -[c - \lambda_1(t)]V(t, X(t)) + \lambda_2(t)V^\beta(t, X(t)) \quad (0 \leq \beta < 1), \)

where \( c > 0 \) is a constant and \( \lambda_i(t) > 0 \) \((i = 1, 2)\) are continuous functions satisfying

\[
\limsup_{t \rightarrow +\infty} \frac{1}{V(t)} \int_{t}^{t+\delta} \lambda_1(s)ds < c \quad (2.3)
\]

and

\[
\sup_{t \geq 0} \int_{t}^{t+1} \lambda_2(s)ds < \infty. \quad (2.4)
\]

Then the solutions of \((2.1)\) are uniformly ultimately bounded.

**Proof.** See Lemma 2.1 in [13] for \( \beta = \frac{1}{2} \). \( \square \)

3. Main results

**Theorem 3.1.** Suppose that \( a, b, b_1, c, \delta_0 \) are positive constants, \( p = p(t) \) and that

(i) \( h(0) = 0, \delta_0 \leq h(x)/x \), for all \( x \neq 0 \);

(ii) \( h(x) \leq c \) for all \( x \);

(iii) \( b \leq g(y)/y \leq b_1 \), for all \( y \neq 0 \);

(iv) \( a \leq f(z)/z \), for all \( z \neq 0 \);

(v) \( \int_{0}^{\infty} p(\mu)d\mu \leq P_0 < \infty \) where \( P_0 \) is a positive constant.

Then for any given finite constants \( x_0, y_0, z_0 \) there exists a constant \( D = D(x_0, y_0, z_0) \), such that any solution \((x(t), y(t), z(t))\) of the system \((1.2)\) determine by \( x(0) = x_0, y(0) = y_0, z(0) = z_0 \) for \( t = 0 \), satisfies

\[
|x(t)| \leq D, \quad |y(t)| \leq D, \quad |z(t)| \leq D, \quad (3.1)
\]

for all \( t \geq 0 \).

**Remark 3.2.** When \( f(\ddot{x}) = a\ddot{x}, \ g(\dot{x}) = b\dot{x}, \ h(x) = cx \) and \( p(t, x, \dot{x}, \ddot{x}) = 0 \), Eq. \((1.1)\) reduces to a linear constant coefficient differential equation and conditions \((i)-(v)\) of Theorem 3.1 reduce to the corresponding Routh–Hurwitz criterion \( a > 0, \ ab > c \) and \( c > 0 \).
The proofs of Theorem 3.1 and subsequent results depend on some certain fundamental properties of a continuously differentiable function \( V(t) = V(x(t), y(t), z(t)) \) defined by

\[
2V(t) = 2a \int_0^t h(\xi)d\xi + 2 \int_0^t g(\tau)d\tau + 2yh(x) + zbx^2 + (\alpha + a^2)y^2 + z^2 + 2axay + 2axz + 2ayz,
\]

where \( \alpha \) is a positive fixed constant satisfying

\[
0 < \alpha < b - \frac{c}{a}.
\]

The Eq. (3.2) and its time derivatives satisfy some fundamental inequalities as will be seen later. In what follows, we shall state and prove some results that would be useful in the proof of the main result.

**Lemma 3.3.** Under the hypotheses of Theorem 3.1, there exist positive constants \( D_i (i = 0, 1) \) such that for all \( (x, y, z) \in \mathbb{R}^3 \)

\[
D_0(x^2(t) + y^2(t) + z^2(t)) \leq V(t) \leq D_1(x^2(t) + y^2(t) + z^2(t)).
\]

**Proof.** We observe that the function in Eq. (3.2) can be rewritten as

\[
2V(t) = V_1 + V_2,
\]

where

\[
V_1 = 2a \int_0^t h(\xi)d\xi + 2 \int_0^t g(\tau)d\tau + 2yh(x)
\]

and

\[
V_2 = zbx^2 + (\alpha + a^2)y^2 + z^2 + 2axay + 2axz + 2ayz.
\]

In view of hypothesis (iii) in Theorem 3.1, \( g(y) \geq by \) for all \( y \neq 0 \), thus

\[
2 \int_0^t g(\tau)d\tau + 2yh(x) \geq (by + h(x))t - b^{-1}h^2(x) \geq -b^{-1}h^2(x).
\]

This is true since \( (by + h(x))^2 \geq 0 \) for all \( x, y \). Moreover, hypotheses (i) and (ii) of Theorem 3.1 imply that

\[
2a \int_0^t h(\xi)d\xi = 2b^{-1} \int_0^t (ab - h'(\xi))h(\xi)d\xi + b^{-1}h^2(x) \geq (ab - c)b^{-1} \delta_0 x^2 + b^{-1}h^2(x).
\]

On combining the inequalities (3.5) and (3.6), we obtain

\[
V_1 \geq (ab - c)b^{-1} \delta_0 x^2
\]

for all \( x \). Furthermore, \( V_2 \) can be rewritten as

\[
V_2 = XQ_0 x^3,
\]

where \( X = (x \ y \ z), Q_0 = \begin{pmatrix} \frac{ab}{a} & \frac{\alpha a}{a} & \frac{\alpha}{a} \\ \frac{ab}{a} & \frac{\alpha + a^2}{a} & \frac{\alpha}{a} \\ \frac{\alpha}{a} & \frac{\alpha}{a} & 1 \end{pmatrix} \) and \( \det Q_0 = x^2(b - \alpha) > 0 \), since \( b - \alpha > 0 \) (which follows from (3.3)). Thus

\[
V_2 \geq \alpha^2(x^2 + y^2 + z^2)^2
\]

for all \( (x, y, z) \in \mathbb{R}^3 \) with \( \alpha > 0 \). On gathering the inequalities (3.7) and (3.8), the lower inequality in (3.4) is obtained. Now to obtain the upper inequality in (3.4), we proceed as follows. Since \( h(0) = 0 \), hypothesis (ii) of the Theorem 3.1 implies that \( h(x) \leq \alpha x \) for all \( x \neq 0 \). It follows from hypotheses (ii) and (iii) of the theorem that

\[
V_1 \leq c(a + 1)x^2 + (b_1 + c)y^2,
\]

\[
V_2 \leq \alpha(a + b + 1)x^2 + (\alpha + a)(a + 1)y^2 + (\alpha + a + 1)z^2.
\]

On gathering the estimates (3.9) and (3.10), the upper inequality in (3.4) follows immediately. \( \square \)

From (3.2) it is clear that \( V(0, 0, 0) = 0 \), the lower inequality in the inequalities (3.4) implies, \( V(x, y, z) > 0 \) as \( x^2 + y^2 + z^2 \neq 0 \), hence it follows that

\[
V(x, y, z) \rightarrow \infty \quad \text{as} \quad x^2 + y^2 + z^2 \rightarrow \infty.
\]

Inequality (3.4) together with (3.11) established condition (i) of the Lemma 2.2.

**Lemma 3.4.** Under the hypotheses of the theorem, there are positive constants \( D_i (i = 2, 3, 4, 5) \) such that if \( (x(t), y(t), z(t)) \) is any solution of the system (1.2), then
\[
\dot{V}(t) = \frac{d}{dt} V(x(t), y(t), z(t)) \leq -(D_2 x^2 + D_3 y^2 + D_4 z^2) + D_5 (|x| + |y| + |z|)|p(t)|.
\] (3.12)

**Proof.** Along any solution \((x(t), y(t), z(t))\) of the system (1.2), it follows from the Eq. (3.2) that
\[
\dot{V}(t) = -axh(x) - (ayg(y) - y^2 h'(x)) + \alpha(g(y) - by) - (ax + ay + z)(f(z) - az) + (ax + ay + z)p(t) + 2yQ_1Y^T,
\] (3.13)
where \(Y = (y \quad z)\), \(Q_1 = \left(\begin{array}{cc} a & 1 \\ 1 & 0 \end{array}\right)\), and \(\det Q_1 = -1\). In view of hypotheses (i)--(iv), we have that
\[
\dot{V}(t) \leq -\frac{1}{2} \alpha \delta_0 x^2 - \frac{7}{8} (x + ab - c)y^2 - \frac{1}{2} \alpha \delta_0 z^2 - W_1 + (ax + ay + z)p(t) \quad (j = 1, 2, 3),
\] (3.14)
where
\[
W_1 = \alpha \left(\frac{1}{4} \delta_0 x^2 - (g(y) - by)x + \frac{1}{16\alpha} (x + ab - c)y^2\right);
\] (3.15)
\[
W_2 = \alpha \left(\frac{1}{4} \delta_0 x^2 - (f(z) - az)x + \frac{1}{4} z^2\right);
\] (3.16)
\[
W_3 = \alpha \left(\frac{1}{16a} (x + ab - c)y^2 + (f(z) - az)y + \frac{\alpha}{4a} z^2\right).
\] (3.17)
Using the Eqs. (3.15)--(3.17), and taking into consideration the following inequalities
\[
(g(y) - by)^2 < \frac{\delta_0 (x + ab - c)}{16\alpha} y^2;
\] (3.18)
\[
(f(z) - az)^2 < \frac{\delta_0}{4} z^2;
\] (3.19)
\[
(f(z) - az)^2 < \frac{\alpha (x + ab - c)}{16a^2} z^2;
\] (3.20)
we have that
\[
W_1 \geq \frac{\alpha}{16} \left(2\sqrt{\delta_0} |x| - \sqrt{\frac{x + ab - c}{\alpha} y}\right)^2 \geq 0 \quad \text{for all } x, y;
\] (3.21)
\[
W_2 \geq \frac{\alpha}{4} \left(\sqrt{\delta_0} |x| - \sqrt{\frac{y}{a}}\right)^2 \geq 0 \quad \text{for all } x, z;
\] (3.22)
\[
W_3 \geq \frac{a}{16} \left(\sqrt{\frac{x + ab - c}{a}} y - 2 \sqrt{\frac{y}{a}}\right)^2 \geq 0 \quad \text{for all } y, z.
\] (3.23)
On making use of the estimates (3.21)--(3.23) in (3.14), we obtain
\[
\dot{V}(t) \leq -\frac{1}{2} \alpha \delta_0 x^2 - \frac{7}{8} (x + ab - c)y^2 - \frac{1}{2} \alpha \delta_0 z^2 + \max(x, a, 1) (|x| + |y| + |z|)|p(t)|,
\] (3.24)
and this completes the proof of the Lemma 3.4. □

At last we shall now give the proof of the Theorem 3.1.

**Proof of Theorem 3.1.** Let \((x(t), y(t), z(t))\) be any solution of (1.2), then from (3.24), it follows that
\[
\dot{V}(t) \leq \delta_1 (3 + x^2 + y^2 + z^2)|p(t)|,
\]
where \(\delta_1 \equiv \max(x, a, 1)\). Now, from the inequalities (3.4), we obtain
\[
\dot{V}(t) - \delta_2 V(t)|p(t)| \leq \delta_2 |p(t)|
\]
where \(\delta_2 = \max(3\delta_1, \delta_1 D_0^{-1})\). Multiplying each side by the integrating factor \(\exp\left(-\int^t_0 |p(\mu)|d\mu\right)\), and integrate from 0 to \(t\) to obtain
\[
V(t) \leq V(0) e^{\delta_2 t} + e^{\delta_2 t} - 1 \equiv \delta_3 (x_0, y_0, z_0),
\]
where \(\delta_3 = \delta_3 D_0^{-1}\), this verifies the inequalities (3.1) with \(D \equiv \delta_3^{1/2}\). This completes the proof of the Theorem 3.1.
Our next result is about the ultimate boundedness of solutions to the Eq. (1.2). □

**Theorem 3.5.** Suppose that a, b, b₁, c, δ₀ are positive constants and that

(i) Conditions (i)–(iv) of the Theorem 3.1 hold;

(ii) for all (x, y, z) ∈ ℜ³ and 0 ≤ t ∈ ℜ⁺ there are nonnegative continuous functions p₁(t) and p₂(t) such that

\[ |p(t, x, y, z)| \leq p₁(t) + p₂(t)(|x| + |y| + |z|) \quad \text{and} \quad |x| + |y| + |z| \geq \rho \quad (\rho > 0). \quad (3.25) \]

where \( \sup_{t \in [0, \infty)} p₁(t) d\mu < \infty \) and there is \( \varepsilon > 0 \) such that \( 0 < p₂(t) < \varepsilon \).

Then the solution \( (x(t), y(t), z(t)) \) of (1.2) is uniformly ultimately bounded.

**Proof of Theorem 3.5.** Consider the equivalent system (1.2) and the Lyapunov function \( V(t, x, y, z) \) as defined in (3.2). If the inequalities (3.4) hold for \( V(x, y, z) \), it follows that

\[ V(x, y, z) \to \infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \quad (3.26) \]

From the inequalities (3.4) and relation (3.26), condition (ii) of Lemma 2.2 is established.

Next, we shall show that condition (ii) of Lemma 2.2 holds for the system (1.2). To see this, conclusion of Lemma 3.4 can be revised as follows

\[ V(1.2)(t) \leq -\min(D₁, D₂, D₃)(x^2 + y^2 + z^2) + D₅(|x| + |y| + |z|)p₁(t) \]

\[ \leq -\delta₁(x ∧ y + z^2) + D₂(|x| + |y| + |z|)^2p₂(t) + D₅(|x| + |y| + |z|)p₁(t) \]

\[ \leq -\delta₁(x^2 + y^2 + z^2) + 3D₂(x^2 + y^2 + z^2)p₂(t) + \sqrt{3D₅(x^2 + y^2 + z^2)^{1/2}}p₁(t), \]

provided \( |x| + |y| + |z| \geq \rho \). Using the inequalities (3.4), for all \( (x, y, z) \in ℜ³ \) and \( 0 \leq t \in ℜ⁺ \), we have that

\[ V(1.2)(t) \leq \left[ -\delta₁D₁^{-1} - 3D₄^{-1}D₂p₂(t) \right] V(x, y, z) + D₅\sqrt{3D₅}p₁(t). \]

Let

\[ p₂(t) = \lim \sup_{(x, y, z) \to (\infty, \infty)} \frac{1}{t} \int_{t-x}^{t} p₁(\mu) d\mu < 3^{-1}\delta₁^{-1}D₁^{-1}D₂^{-1}. \]

Thus, choose \( \delta = \delta₁D₁^{-1}, \lambda₁(t) = 3D₄^{-1}D₂p₂(t), \lambda₂(t) = D₅\sqrt{3D₅}p₁(t) \) and \( \beta = 1/2 \), condition (ii) of Lemma 2.2 is established. This completes the proof of the Theorem 3.5. □

**References**


[6] J. Andres, Boundedness results for solutions of the equation \( x(t) = ax + g(x)x + h(x) = p(t) \) without the hypothesis \( h(x)|g(x) | \geq 0 \) for \( |x| > |R|, \) Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 80 (8) (1986) 533–539.


